6 Line Segments and Rays

(6.1) Definition (line segment $\overline{AB}$)

If A and B are distinct points in a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ then the line segment from A to B is the set $\overline{AB} = \{M \in \mathcal{S} \mid A - M - B \text{ or } M = A \text{ or } M = B\}$.

1. Let $A(-1/2, \sqrt{3}/2)$ and $B(\sqrt{19}/10, 1/10)$ denote given points of line ${}_0L_1$. Give a graphical sketch for line segment $\overline{AB}$.

2. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ denote three points which belong to the type II line ${}_cL_r$ in the Poincaré Plane. If $x_1 < x_3 < x_2$ show that then $C \in \overline{AB}$.

3. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the type II line ${}_cL_r$ in the Poincaré Plane. If $x_1 < x_2$ show that $\overline{AB} = \{C = (x, y) \in {}_cL_r \mid x_1 \le x \le x_2\}$.

(6.2) Definition

Let $\mathcal{A}$ be a subset of a metric geometry. A point $B \in \mathcal{A}$ is a passing point of $\mathcal{A}$ if there exists points $X, Y \in \mathcal{A}$ with $X - B - Y$. Otherwise B is an extreme point of $\mathcal{A}$.

4. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ denote two points in metric geometry, and let $C \in \overline{AB}$. If $C \neq A$ and $C \neq B$ explain is point C passing point or extreme point of $\overline{AB}$.

(6.3) Theorem

If A and B are two points in a metric geometry then the only extreme points of the segment $\overline{AB}$ are A and B themselves. In particular, if $\overline{AB} = \overline{CD}$ then $\{A, B\} = \{C, D\}$.

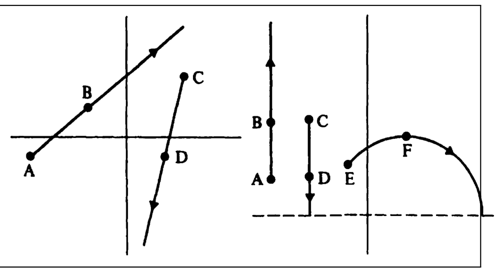
(6.4) Definition (end points, length of the segment $\overline{AB}$)

The end points (or vertices) of the segment $\overline{AB}$ are A and B. The length of the segment $\overline{AB}$ is $AB = d(A, B)$.

(6.5) Definition (ray $pp[A, B) = \overrightarrow{AB}$)

If A and B are distinct points in a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ then the ray from A toward B is the set

$$pp[A, B) = \overrightarrow{AB} = \overline{AB} \cup \{C \in \mathcal{S} \mid A - B - C\}.$$



(6.6) Theorem

In a metric geometry (i) *if $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$;* (ii) *if $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$.*

(6.7) Definition (vertex of the ray)

The vertex (or initial point) of the ray $pp[A, B) = \overrightarrow{AB}$ is the point A.

(6.8) Theorem

If A and B are distinct points in a metric geometry then there is a ruler $f : \overleftrightarrow{AB} \to \mathbb{R}$ such that $pp[A, B) = \overrightarrow{AB} = \{X \in \overleftrightarrow{AB} \mid f(X) \ge 0\}$

(6.9) Definition ($\overline{AB} \cong \overline{CD}$)

Two line segments $\overline{AB}$ and $\overline{CD}$ in a metric geometry are congruent (written $\overline{AB} \cong \overline{CD}$) if their lengths are equal; that is $\overline{AB} \cong \overline{CD}$ if $AB = CD$.

(6.10) Theorem (Segment Construction)

If $\overrightarrow{AB}$ is a ray and $\overline{PQ}$ is a line segment in a metric geometry, then there is a unique point $C \in \overrightarrow{AB}$ with $\overline{PQ} \cong \overline{AC}$.

5. Prove Theorems 6.3, 6.6, 6.8 and 6.10.

6. In the Poincaré Plane let $A(0, 2)$, $B(0, 1)$, $P(0, 4)$, $Q(7, 3)$. Find $C \in \overrightarrow{AB}$ so that $\overline{AC} \cong \overline{PQ}$.

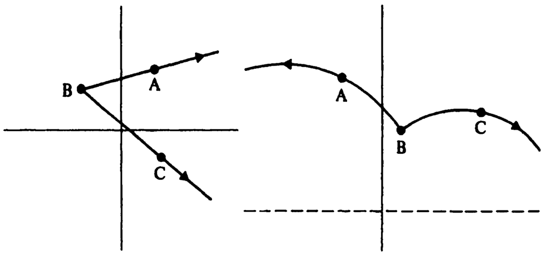
7. Let A and B be distinct points in a metric geometry. Then $M \in \overleftrightarrow{AB}$ is a midpoint of the line segment $\overline{AB}$ if and only if $AM = MB$. (Remember that here AM means $d(A, M)$.) (a) If M is a midpoint of $\overline{AB}$, prove that $A - M - B$. (b) Show that $\overline{AB}$ has a midpoint M, and that M is unique. (c) Let $A(0, 9)$ and $B(0, 1)$. Find the midpoint of $\overline{AB}$ where A and B are points of (i) the Euclidean plane; (ii) the Hyperbolic plane.

8. Determine are the statements true or false: (a) $\overline{AB} = \overline{CD}$ only if $A = C$ or $A = D$. (b) If $AB = CD$ then $A = C$ or $A = D$. (c) If $\overline{AB} \cong \overline{CD}$, then $\{A, B\} = \{C, D\}$. (d) If $\overline{AB} \cong \overline{CD}$, then $\overline{AB} = \overline{CD}$. (e) A point on $\overleftrightarrow{AB}$ is uniquely determined by its distances from A and B.

9. In a metric geometry $(\mathcal{S}, \mathcal{L}, d)$, prove that if $A - B - C$, $P - Q - R$, $\overline{AB} \cong \overline{PQ}$, $\overline{AC} \cong \overline{PR}$, then $\overline{BC} \cong \overline{QR}$.

7 Angles and Triangles

It is important to note that an angle is a set, not a number like 45°. We will view numbers as properties of angles when we define angle measure in section: "The Measure of an Angle".



(7.1) Definition (angle $\measuredangle ABC$)
If A, B and C are noncollinear points in a metric geometry then the angle $\measuredangle ABC$ is the set

$$\measuredangle ABC = \overrightarrow{BA} \cup \overrightarrow{BC} = pp[B, A) \cup pp[B, C).$$

1. Show that B is not a passing point of $\measuredangle ABC$.

(7.2) Lemma
In a metric geometry, B is the only extreme point of $\measuredangle ABC$.

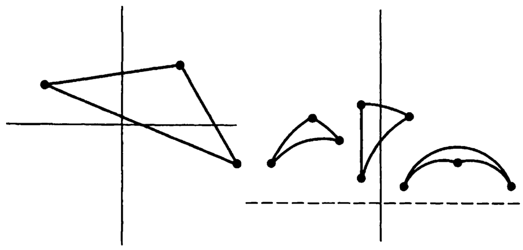
(7.3) Theorem ($\measuredangle ABC = \measuredangle DEF \Rightarrow B = E$)
In a metric geometry, if $\measuredangle ABC = \measuredangle DEF$ then $B = E$.

2. Prove Lemma 7.2 and Theorem 7.3.

(7.4) Definition (vertex of the angle $\measuredangle ABC$)
The vertex of the angle $\measuredangle ABC$ in a metric geometry is the point B.

(7.5) Definition (triangle $\triangle ABC$)
If $\{A, B, C\}$ are noncollinear points in a metric geometry then the triangle $\triangle ABC$ is the set

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}.$$



(7.6) Lemma
In a metric geometry, if A, B, and C are not collinear then A is an extreme point of $\triangle ABC$.

(7.7) Theorem
In a metric geometry, if $\triangle ABC = \triangle DEF$ then $\{A, B, C\} = \{D, E, F\}$.

3. Prove Lemma 7.6 and Theorem 7.7.

(7.8) Definition (vertices, sides)
In a metric geometry the vertices of $\triangle ABC$ are the points A, B, C. The sides (or edges) of $\triangle ABC$ are $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

4. Prove that $\measuredangle ABC = \measuredangle CBA$ in a metric geometry.

In next three problems do not use Lemma 7.6 and Theorem 7.7.

5. Let D, E and F be three noncollinear points of a metric geometry and let ℓ be a line that contains at most one of D, E and F. Prove that each of $\overleftrightarrow{DE}$, $\overleftrightarrow{DF}$ and $\overleftrightarrow{EF}$ intersects ℓ in at most one point.

6. Prove that if $\triangle ABC = \triangle DEF$ in a metric geometry then $\overleftrightarrow{AB}$ contains exactly two of the points D, E and F.

7. In a metric geometry, prove that if A, B and C are not collinear then $\overline{AB} = \overleftrightarrow{AB} \cap \triangle ABC$.